Math 53: Multivariable Calculus **Sections 201, 204**

Worksheet answers for 2021-10-06

If you would like clarification on any problems, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to conceptual questions

Question 1. No, for instance there does not exist any function $f(x, y)$ with partials $f_x(x, y) = -y$ and $f_y(x, y) = x$. This is because Clairaut's theorem would be violated.

However, satisfying Clairaut's theorem is not quite good enough: the functions

$$\frac{-y}{x^2 + y^2}, \; \frac{x}{x^2 + y^2}$$

are defined on $\mathbb{R}^2$ minus the origin, but there does not exist a function f defined on that region with these partial derivatives.

Question 2. ∇f and ∇g can either be in the same direction or opposite directions.

Question 3. No, because D could be negative. However, there is enough information to eliminate the possibility that P is a local maximum (it must be a local minimum or a saddle point).

Question 4. Yes, it does. Intuitively, it's saying that the function is a concave up in every direction (if we take $\mathbf{u} = \langle 1, 0 \rangle$ for example, we just get $f_{xx}(P) > 0$, showing that it's concave up in the x-direction).

In fact the written condition is equivalent to the 2nd derivative test for a local minimum (but this is tricky to show).

For the interested, if we write H_f for the *Hessian matrix*

$$H_f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

the stated inequality is equivalent to

$$\mathbf{u}^\top H_f(P)\mathbf{u} > 0$$

for all unit (column) vectors $\mathbf{u}$, which in linear algebra language is the condition that $H_f(P)$ is *positive definite*. The second derivative test that you are familiar with is just Sylvester's criterion for positive definiteness applied to $H_f(P)$.

Question 5. Sure. Take $f(x, y) = xy$ for example, or a more fancy example could be $f(x, y) = y^2 - x^2(x + 1)$.

The gradient at the point of self-intersection is the zero vector, so it is not possible to write down a tangent line using that method.

Question 6. The point (a, b, c) will also give a solution to the Lagrange multipliers system of equations, *however* it may not be a maximum. In fact, there is no guarantee that the new problem will even have an absolute maximum or minimum at all!

If appropriate additional hypotheses are imposed (such as in the field of convex optimization), better things can be said about this so-called "dual problem." But that's beyond the scope of this course (and not something I have any expertise in).